On the Entropy of a Quantum Field in the 2 + 1 Dimensional Spinning Black Holes

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Abstract

We calculate the entropy of a scalar field in a rotating black hole in 2 + 1 dimension. In the Hartle-Hawking state the entropy is proportional to the horizon area, but diverges linearly in $\sqrt{h}$, where h is the radial cut-off. In WKB approximation the superradiant modes do not contribute to the entropy.

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Recently, many efforts have been concentrated on understanding the statistical origin of the Bekenstein-Hawking black hole entropy [1]: the brick wall method of 't Hooft [2], the entanglement entropy [3], the conical method [4], etc. (See the review [5].) The common property of the above methods is that the entropy is divergent and proportional to the horizon area.

For a rotating black hole in 4 dimensional space-time the entropy of a quantum field was calculated by the brick wall method [6]. The result is that the entropy is proportional to the horizon area in the Hartle-Hawking state. The difficulty in treating the quantum field in a rotating black hole background is that one can not find a global static frame. Usually one resolve it by taking a rigid frame co-rotating with the black hole. However in this case an observer who is at the outside of a surface (the velocity of light surface (VLS)) must have $v \geq 1$ and must move on a spacelike world line. To remove such an unphysical behavior one needs a perfectly reflecting mirror inside the VLS [7].

In 3 dimension Banados, Teitelboim, and Zanelli (BTZ) obtained a black hole solution for the standard $2 + 1$ Einstein-Maxwell theory with a negative cosmological constant, which (for charge $= 0$) is asymptotically anti-de Sitter space-time [8]. This is also the solution of the low energy string action in 3 dimension [9]. Chan and Mann modified the BTZ black hole and obtained a new class of spinning black hole solutions [10]. The black hole is characterized by mass, angular momentum, and charge, which is similar to the 4 dimensional rotating black hole. Therefore to study the 3 dimensional black hole is helpful to understand the entropy of the 4 dimensional black hole.

The entropy of the BTZ black hole was calculated by the brick wall method in Ref. [11]. They found that the entropy is finite for the BTZ black hole with non-zero angular momentum. In this paper we study the entropy of a quantum field in 3 dimensional spinning black hole [10] by the brick wall method. We show that the entropy diverges linearly in $\sqrt{h}$, where h is the radial coordinate distance from the horizon to the brick wall. In WKB approximation the superradiant modes in the Hartle-Hawking state do not contribute to the entropy.

Let us consider a scalar field with mass μ in thermal equilibrium at temperature $1/\beta$ in a rotating

3 dimensional black hole background, of which line element is generally given by

$$ds^2 = g_{tt}(r)dt^2 + 2g_{t\phi}(r)dtd\phi + g_{\phi\phi}(r)d\phi^2 + g_{rr}(r)dr^2. \tag{1}$$

This metric has two Killing vector fields: the timelike Killing vector $\xi^\mu = (\partial_t)^\mu$ and the axial Killing vector $\psi^\mu = (\partial_\phi)^\mu$. In this paper we consider the spinning black hole with the following metric components [10]

$$
\begin{aligned}
g_{tt} &= -\left(\frac{8\Lambda\alpha^2}{(3N-2)N}r^N + Ar^{1-\frac{N}{2}}\right), \\
g_{t\phi} &= -\frac{\omega}{2}r^{1-\frac{N}{2}}, \\
g_{\phi\phi} &= \left(\alpha^2 r^N - \frac{\omega^2}{4A}r^{1-\frac{N}{2}}\right), \\
g_{rr} &= \alpha^2\left[\frac{8\Lambda\alpha^2}{(3N-2)N}r^N + \left(A - \frac{2\Lambda\omega^2}{(3N-2)NA}\right)r^{1-\frac{N}{2}}\right]^{-1},
\end{aligned}
\tag{2}
$$

where A and ω are integration constants and α is a length scale with dimensions of length. The mass and the angular momentum of the black hole is given by

$$
M = \frac{N}{2}\left[\frac{2\Lambda\omega^2}{(3N-2)NA}\left(\frac{4}{N}-3\right) - A\right], \tag{3}
$$

$$
J = \frac{3N-2}{4}\omega. \tag{4}
$$

The black hole exist if $\Lambda > 0$ and $2 \geq N > \frac{2}{3}$. The constant A is negative so that $g_{\phi\phi} > 0$. For this spinning black hole there are two important surfaces: the outer horizon and the stationary limit surface. The outer horizons are given by

$$
r_+^{\frac{3N}{2}-1} = \left(\frac{8\Lambda\alpha^2}{(3N-2)N}\right)^{-1}\left[-A + \frac{2\Lambda\omega^2}{(3N-2)NA}\right], \tag{5}
$$

and the stationary limit surface is given by

$$
r_s^{\frac{3N}{2}-1} = \left(\frac{8\Lambda\alpha^2}{(3N-2)N}\right)^{-1}(-A). \tag{6}
$$

The Killing vector ξ^μ vanishes on the stationary limit surface, and the Killing vector $\xi^\mu + \Omega_H\psi^\mu$ is null on the event horizon ($r = r_+ = r_H$), where Ω_H is the angular velocity of the horizon [13]:

$$
\Omega_H = \lim_{r \to r_H}\left(-\frac{g_{t\phi}}{g_{\phi\phi}}\right) = -\frac{4\Lambda\omega}{(3N-2)NA}. \tag{7}
$$

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When $N = 2$, this black hole solution reduces to the spinning BTZ one. When $N = 1$, this solution becomes the modification of the black hole of Mandal, Sengupta, and Wadia [12].

The equation of motion of the field with mass μ is given by

$$\left[\nabla_\mu \nabla^\mu - \xi R - \mu^2\right] \Psi = 0, \tag{8}$$

where ξ is an arbitrary constant and $R(x)$ is the scalar curvature. $\xi = 1/8$ and $\mu = 0$ case corresponds to the conformally coupled one. We assume that the scalar field is rotating with a constant azimuthal angular velocity $\Omega_0 \leq \Omega_H$. The associated conserved quantity is angular momentum. The positive frequency field mode can be written as $\Phi_{q,m} = f_{q,m}(r)e^{-i\mathcal{E}t+im\phi}$, where m is the azimuthal quantum number and q denotes other quantum numbers. The free energy of the system is then given by

$$F = \frac{1}{\beta} \sum_{j,m} d_{j,m} \ln \left(1 - e^{-\beta(\mathcal{E}_{j,m}-m\Omega_0)}\right) \tag{9}$$

or

$$F = \frac{1}{\beta} \sum_m \int_0^\infty d\mathcal{E} g(\mathcal{E}, m) \ln \left(1 - e^{-\beta(\mathcal{E}-m\Omega_0)}\right), \tag{10}$$

where $g(\mathcal{E}, m)$ is the density of state for a given $\mathcal{E}$ and m.

To evaluate the free energy we follow the brick wall method of 't Hooft [2]. We impose a small radial cut-off h such that

$$\Psi(x) = 0 \quad \text{for} \quad r \leq r_H + h, \tag{11}$$

where r_H denotes the coordinate of the event horizon. To remove the infra-red divergence we also introduce another cut-off $L \gg r_H$ such that

$$\Psi(x) = 0 \quad \text{for} \quad r \geq L. \tag{12}$$

In the WKB approximation with $\Psi = e^{-i\mathcal{E}t+im\phi+iS(r)}$ the equation (8) yields the constraint [15]

$$p_r^2 = \frac{1}{g^{rr}} \left[-g^{tt}\mathcal{E}^2 + 2g^{t\phi}\mathcal{E}m - g^{\phi\phi}m^2 - V(x)\right], \tag{13}$$

where $p_r = \partial_r S$ and $V(x) = \xi R(x) + \mu^2$. It is important to note that the number of state for a given $\mathcal{E}$ is determined by p_r and m. The number of mode with energy less than $\mathcal{E}$ and with a fixed m is

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obtained by integrating over the phase space

$$
\begin{aligned}
\Gamma(\mathcal{E}, m) &= \frac{1}{\pi} \int d\phi \int dr p_r(\mathcal{E}, m, x) \\
&= \frac{1}{\pi} \int d\phi \int dr \left[\frac{1}{g^{rr}} \left(-g^{tt}\mathcal{E}^2 + 2g^{t\phi}\mathcal{E}m - g^{\phi\phi}m^2 - V(x) \right) \right]^{\frac{1}{2}}.
\end{aligned}
\tag{14}
$$

At this point we need some remarks. In a rotating system, in general, there is a superradiance effect, which occurs when $0 < \mathcal{E} < m\Omega_0$. For this range of the frequency the free energy F becomes a complex number. In case $\mathcal{E} = m\Omega_0$ the free energy is divergent. Therefore to obtain a real finite value for the free energy F, we must require that $\mathcal{E} > m\Omega_0$. (For $0 < \mathcal{E} < m\Omega_0$ the free energy diverges. See below.) This requirement says that we must restrict the system to be in the region such that $g'_{tt} \equiv g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0$. In this region the free energy is a finite real value because $\mathcal{E} - m\Omega_0 > 0$. It is easily shown as follows. Let us define $E = \mathcal{E} - m\Omega_0$. Then it is written as

$$
\begin{aligned}
E &= \left(\frac{g^{t\phi}}{g^{tt}} - \Omega_0 \right) m + \frac{1}{-g^{tt}} \left[\left(g^{t\phi}m \right)^2 + \left(-g^{tt} \right) \left(V + g^{\phi\phi}m^2 + g^{rr}p_r^2 \right) \right]^{1/2} \\
&= (\Omega - \Omega_0) m + \frac{-\mathcal{D}}{g_{\phi\phi}} \left[\frac{1}{-\mathcal{D}} m^2 + \frac{g_{\phi\phi}}{-\mathcal{D}} \left(V + \frac{p_r^2}{g_{rr}} \right) \right]^{1/2},
\end{aligned}
\tag{15}
$$

where we used

$$
g^{tt} = \frac{g_{\phi\phi}}{\mathcal{D}}, \quad g^{t\phi} = \frac{-g_{t\phi}}{\mathcal{D}}, \quad g^{\phi\phi} = \frac{g_{tt}}{\mathcal{D}},
\tag{16}
$$

and $\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}}$. Here $-\mathcal{D} = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$. From Eq.(15), for all m and p_r one can see that the condition such that $E > 0$ is

$$
\frac{\sqrt{-\mathcal{D}}}{g_{\phi\phi}} \pm (\Omega - \Omega_0) > 0
\tag{17}
$$

or

$$
g'_{tt} \equiv g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0.
\tag{18}
$$

Therefore in the region such that $-g'_{tt} > 0$ (called region I) the free energy is real, but in the region such that $-g'_{tt} < 0$ (called region II) the free energy is complex. However in the region II the integration over the momentum phase space is divergent. This fact becomes apparent if we investigate

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the momentum phase space. In the region I the points of p_i satisfying $\mathcal{E} - \Omega_0 p_\phi = E$ for a given E are located on the following curve

$$\frac{p_r^2}{g_{rr}} + \frac{-g'_{tt}}{-\mathcal{D}} \left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}} E \right)^2 = \left(\frac{E^2}{-g'_{tt}} - V \right), \tag{19}$$

which is the ellipse, *a closed curve*. Here $p_\phi = m$. So the density of state $g(E)$ for a given E is finite and the integrations over p_i give a finite value. But in the region II the points of p_i are located on the following curve

$$\frac{p_r^2}{g_{rr}} - \frac{g'_{tt}}{-\mathcal{D}} \left(p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}} E \right)^2 = - \left(\frac{E^2}{g'_{tt}} + V \right), \tag{20}$$

which is the hyperbola, *a open curve*. So $g(E)$ diverges and the integrations over p_i diverge. In case of $g'_{tt} = 0$, the points of p_i are given by

$$\frac{p_r^2}{g_{rr}} = \frac{p_\phi - \left(\frac{g_{\phi\phi}E^2}{\mathcal{D}} + V \right) / \left(\frac{2g_{t\phi}}{\mathcal{D}} E \right)}{\frac{-\mathcal{D}}{2g_{t\phi}E}}, \tag{21}$$

which is a parabola and also *open curve*. Therefore the value of the p_i integrations are divergent. Actually the surface (the curve) such that $g'_{tt} = 0$ is the velocity of the light surface (VLS). Beyond VLS (in region II) the co-moving observer must move more rapidly than the velocity of light. It is unphysical. Thus we assume that the system is in the region I.

Now let us determine the region I. From

$$\begin{aligned} g'_{tt} &= g_{tt} + 2\Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} \\ &= -\left(\frac{8\Lambda\alpha^2}{(3N-2)N} - \Omega_0^2 \alpha^2 \right) r^N + \left(-A - \Omega_0\omega - \Omega_0^2 \frac{\omega^2}{4A} \right) r^{1-\frac{N}{2}} \end{aligned} \tag{22}$$

we obtain the exact position of the VLS, which is given by

$$r_{VLS}^{\frac{3N}{2}-1} = \left[\frac{8\Lambda\alpha^2}{(3N-2)N} - \Omega_0^2 \alpha^2 \right]^{-1} \left[-A - \Omega_0\omega - \Omega_0^2 \frac{\omega^2}{4A} \right]. \tag{23}$$

For $\Omega_0 = 0$ the VLS is at $r = r_s$, and for $\Omega_0 = \Omega_H$ it locates at $r = r_+$. As the value of Ω_0 increases from 0 to Ω_H the VLS is continuously moved from r_s to r_+. But there is no outer VLS, which is distinct from the 4-dimensional black hole [6]. Thus the region I is $r_{VLS} < r < \infty$.

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With the assumption that the system is in the region I we obtain the free energy as follows:

$$
\begin{aligned}
\beta F &= \sum_m \int_{m\Omega_0}^{\infty} d\mathcal{E}\, g(\mathcal{E}, m) \ln\left(1 - e^{-\beta(\mathcal{E} - m\Omega_0)}\right) \\
&= \int_0^{\infty} d\mathcal{E} \sum_m g(\mathcal{E} + m\Omega_0, m) \ln\left(1 - e^{-\beta\mathcal{E}}\right) \\
&= -\beta \int_0^{\infty} d\mathcal{E}\, \frac{1}{e^{\beta\mathcal{E}} - 1} \int dm \Gamma(\mathcal{E} + m\Omega_0, m),
\end{aligned}
\tag{24}
$$

where we have integrated by parts and we assume that the quantum number m is a continuous variable. The integration over m yields

$$
F = -\int d\phi \int_{r_H+h}^{L} dr \int_{V(x)\sqrt{-g'_{tt}}}^{\infty} d\mathcal{E}\, \frac{1}{e^{\beta\mathcal{E}} - 1} \frac{\sqrt{g_3}}{\sqrt{-g'_{tt}}} \left(\frac{\mathcal{E}^2}{-g'_{tt}} - V(x)\right).
\tag{25}
$$

In particular when $\Omega_0 = 0$, $J = 0$, and $V(x) = 0$, the free energy (25) is proportional to the volume of the optical space [14]. It is easy to see that the integrand diverges as $r_H + h$ approaches r_{VLS}. In that case the contribution of the $V(x)$ can be negligible.

In the case of $V = 0$ the free energy reduces to

$$
\beta F = -\frac{c}{\beta^2} \int d\phi \int_{r_H+h}^{L} dr \frac{\sqrt{g_3}}{(-g'_{tt})^{3/2}} = -c \int_0^{\beta} d\tau \int d\phi \int_{r_H+h}^{L} dr \sqrt{g_3} \frac{1}{\beta_{local}^3},
\tag{26}
$$

where $\beta_{local} = \sqrt{-g'_{tt}}\,\beta$ is the reciprocal of the local Tolman temperature [17] in the comoving frame, and c is a constant . This form is just the free energy of a gas of massless particles at local temperature $1/\beta_{local}$ in 3 dimension.

From this expression (26) it is easy to obtain the expression for the entropy S of a scalar field for $V(x) = 0$. In the Hartle-Hawking state ($\Omega_0 = \Omega_H, T = T_H$), where

$$
T_H = \frac{\Lambda\alpha^2}{\pi N} \frac{r_H^{\frac{3N}{2}-1}}{\sqrt{g_{\phi\phi}(r_H)}} = \frac{1}{\beta_H}
\tag{27}
$$

the entropy for small h is given by

$$
\begin{aligned}
S &= \left. \beta^2 \frac{\partial}{\partial\beta} F \right|_{\beta=\beta_H, \Omega_0=\Omega_H} \\
&= \frac{6\pi c\alpha^2}{\beta_H^2} \int_{r_H+h}^{L} dr \frac{r^{\frac{N}{2}}}{\left[\left(\frac{8\Lambda\alpha^2}{(3N-2)N} - \Omega_H^2\alpha^2\right) r^N - \left(-A - \Omega_H\omega - \Omega_H^2 \frac{\omega^2}{4A}\right) r^{1-\frac{N}{2}}\right]^{\frac{3}{2}}} \\
&\approx \frac{3c\pi}{2\alpha^4\beta_H^2} \left(\frac{N}{\Lambda}\right)^{\frac{3}{2}} r_H^{\frac{3}{2}-\frac{5}{2}N} g_{\phi\phi}^{\frac{3}{2}}(r_H) \frac{1}{\sqrt{h}} + O(\sqrt{h}).
\end{aligned}
\tag{28}
$$

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The entropy is linearly divergent in $\sqrt{h}$. (This is the general feature of the non-degenerated $2 + 1$ dimensional black hole.) In terms of the proper distance cut-off ϵ the entropy is given by

$$S = \frac{3c}{8\pi^2} \frac{A_H}{\epsilon}, \tag{29}$$

where

$$A_H = 2\pi\sqrt{g_{\phi\phi}(r_H)} = 2\pi \left(\alpha^2 r_H^N - \frac{\omega^2}{4A} r_H^{1-\frac{N}{2}} \right)^{\frac{1}{2}} \tag{30}$$

and

$$\begin{aligned}
\epsilon &= \int_{r_H}^{r_H+h} dr \sqrt{g_{tt}(r)} \\
&\approx \left(\frac{N}{\Lambda r_H^{N-1}} \right)^{\frac{1}{2}} \sqrt{h}.
\end{aligned} \tag{31}$$

Notice that the entropy S (29) does not depends on the constants α, N, A, and ω. In 4 dimensional black hole we also showed that the leading behavior of the entropy of the quantum field is proportional to the horizon area and is not depend on the parameters if we use the proper distance cut-off ϵ [6]. It seems that this is a generic feature of the entropy. In Ref. [11] they argued that the reality condition of the radial mode momentum is that the brick wall must be outside of the stationary limit surface and they said that the entropy is finite. In this paper, we demand that the free energy is finite. This condition is satisfied if the brick wall is present outside of the horizon. In this case, the entropy diverges as the cut-off go to zero.

Let us summarize our result. We have calculated the entropy of the scalar field in the 2+1 dimensional spinning black hole space-time. For the massless field in the Hartle-Hawking state ($\Omega_0 = \Omega_H$ and $\beta = \beta_H$), only in this case, the entropy is proportional to the horizon area, but becomes divergent linearly in $\sqrt{h}$ as the brick wall approaches the horizon. The origin of the divergence is that the momentum phase volume for a given E diverges on the horizon. For the extreme BTZ black hole $(r_+ = r_-)$ (N = 2 case) , $g'_{tt}(r)|_{\Omega_0=\Omega_H} = 0$. So we can not consider the extreme one.

Why there is no the outer velocity of light surface in spinning $2 + 1$ dimensional black hole? In 4 dimensional black hole the outer VLS exists, which show the pathology of the rigid rotation of the

frame. The 4 dimensional black hole space-time is asymptotically flat and non-rotating. But the space-time, for example, of the BTZ black hole is asymptotically anti-de Sitter and have a rotation.

$$ds^2_{BTZ} \stackrel{r \to \infty}{\longrightarrow} -\left(\frac{r^2}{l^2} - M\right) dt^2 - Jdtd\phi + \frac{1}{\frac{r^2}{l^2} - M}dr^2 + r^2d\phi^2. \tag{32}$$

The $g_{t\phi}$ does not vanish at the infinity. It is a constant. Such a fact seems to be a cause of the non-existence of the outer VLS.

Acknowledgment

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